International Journal of Theoretical Physics, Vol. 37, No. 4, 1998

Supermulticonformal Field Theory

A. Ouarab,[1] E. H. El Kinani[1,2] and M. Zakkari[3]

Received October 10, 1997

In this paper we develop the superspace structure of the multicomplex space $M\mathbb{C}_n$ for $n = 2^n$. We extend the basic properties of the multicomplex analysis to the case of $SM\mathbb{C}_n^N$, called supermulticomplex spaces: this goes from the superanalyticity condition to the residue theorem. The formalism of 2D superconformal field theory is also developed on $SM\mathbb{C}_n^{\hat{N}}$. We then show that the associated superconformal symmetry is infinite dimensional and leads to n copies of super-Virasoro algebra. This results can be applied to construct a free-field theory on the volume of the $(n-1)$-super-brane. A model of field theory describing a bosonic case is also presented.

1. INTRODUCTION

Number theory has always been a rich subject, especially when numbers are described in terms of algebra. Recall that only four integer and composition algebras can be constructed, the so-called real, complex, quaternionic (Hamilton, 1848, 1853) and octonionic (Young, 1848) numbers. This is a consequence of the Hurwitz theorem (Hurwitz, 1898; Cayley, 1849). However, by relaxing at least one of the two conditions of validity of the Hurwitz theorem, some extensions have been proposed (Clifford, 1878; Lipschit, 1880; Grassmann, 1855; Graves (1847, 1848). In fact all the possible extensions of complex numbers can be understood in the framework of systems of complex numbers. Among these systems we consider the one generated by a fundamental unit e satisfying the basic relation $e^n = -1$, $n \geq 2$. This system was considered by Weierstrass (1884). The algebra obtained, called

[1] Faculté des Sciences, Département de Physique, LPT BP 1014 Rabat, Morocco; e-mail: ouarab@maghrebnet.net.ma
[2] Faculté des Sciences et Théchniques, Département de Mathématique, BP 509 Errachidia, Morocco.
[3] Faculté des Sciences et Théchniques, Département de Mathématique et Informatique, BP. 577 Settat, Morocco.

multicomplex algebra (Fleury *et al.*, 1993) and denoted $M\mathbb{C}_n$, is an n-dimensional $\mathbf{R}$-algebra given by

$$M\mathbb{C}_n \equiv \left\{ z = \sum_{i=0}^{n-1} x_i \, e^i, \; x_i \in \mathbf{R} \right\} \tag{1.1}$$

It has been shown that most theorems of complex analysis can be extended to the $M\mathbb{C}_n$ spaces with $n > 2$ (Fleury *et al.*, 1995).

Our work consists in defining a supersymmetric extension of the $M\mathbb{C}_n$ spaces. The superspace obtained $SM\mathbb{C}_n^N$ is a set of points parametrized by the multicomplex numbers $\overset{(l)}{z}$ (the n lth conjugates of z) and the Grassmann coordinates $\overset{(l)}{\theta}_1$, $i = 1, 2, \ldots, N_l$, $l = 0, 1, 2, \ldots, n-1$; N refers then to the set $(N_0, N_1, \ldots, N_{n-1})$; we note that

$$SM\mathbb{C}_n^{\hat{N}} = \{(\overset{(l)}{z}, \overset{(l)}{\theta}_i) \qquad l = 0, 1, 2, \ldots, n-1, \, i = 1, 2, \ldots, N_l\} \tag{1.2}$$

A first remark here is that the number of supersymmetries depends on the direction $\overset{(l)}{z}$, in contrast to the usual 2D case. A second important remark is that the above construction is valid only under the assumption $n = 2^p$, $p \geq 1$. Here we will discuss superanalytic and superconformal transformations on $M\mathbb{C}_n^N$. We then consider superconformal tensors on $SM\mathbb{C}_n^N$ and study a plausible superfield theory, $SM\mathbb{C}_n^N$. For simplicity we study the case $N = (0, \ldots, 1, 0, \ldots, 0)$, 1 on the lth position for $l \in \{0, 1, 2, \ldots, n-1\}$. We obtain then n commuting copies of the $N = 1$ super-Virasoro algebra. Finally, we give the action describing a bosonic theory on $M\mathbb{C}_n$.

2. GENERAL

This section is devoted to the introduction of the set $M\mathbb{C}_n$ of multicomplex numbers and certain of their properties which are useful for the sequel. We also give a brief description of superconformal 2D manifolds, in particular the $N = 2$ supersymmetric extension of the usual complex analysis (Belavin *et al.*, 1984). This case will be useful for the study of the superspace structure of $M\mathbb{C}_n$.

Following Fleury *et al.* (1993) consider a system of complex numbers generated by a fundamental unit e which satisfies the basic relation

$$e^n = -1, \qquad n \in N^* \tag{2.1}$$

To not generate the usual complex numbers, the solution $k = \exp(i\pi/n)$ is excluded. A faithful matrix representation of e is given by the $n \times n$ diagonal matrix $(E_{ij})_{1\le i,j\le n}$,

$$E_{ij} = k^{2i-1}\cdot\delta_{ij}, \qquad i, j = 1, 2, 3, \ldots, n \tag{2.2}$$

The multicomplex algebra $M\mathbb{C}_n$ is an n-dimensional $\mathbf{R}$-algebra generated by the free family $\{1, e, e^2, \ldots, e^{n-1}\}$:

$$M\mathbb{C}_n \equiv \left[z = \sum_{i=0}^{n-1} x_i e^i,\ x_i \in \mathbf{R} \right]$$

The algebra $M\mathbb{C}_n$ was provided by a pseudo-norm

$$\|z\|^n = \det \begin{pmatrix} x_0 & x_1 & & x_{n-1} \\ -x_{n-1} & x_0 & & \\ & \ddots & \ddots & x_1 \\ -x_1 & -x_{n-1} & & x_0 \end{pmatrix} = \det \Delta(z) \tag{2.3}$$

Among all the $M\mathbb{C}_n$ spaces, the ones for which $n = 2^{\mathrm{p}}$ have a special status (Fleury *et al.*, 1993). In that case, one defines the lth conjugate of a multicomplex number z as follows:

$$\overset{(l)}{z} = \sum_{i=0}^{n-1} x_i e^{i(2l+1)}, \qquad l = 0, 1, 2, \ldots, n - 1 \tag{2.4}$$

This conjugation satisfies

$$\overset{(l+n)}{z} = \overset{(l)}{z}, \qquad \overset{(l)}{z}_1\, \overset{(l)}{z}_2 = \overset{(l)}{\widehat{z_1 z_2}} \tag{2.5}$$

and leads to the following expression for $\|z\|^n$:

$$\|z\|^{\mathrm{n}} = \prod_{l=0}^{n-1} \overset{(l)}{z} \tag{2.6}$$

It is then convenient to introduce the differential operators

$$\overset{(l)}{\partial} = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-ie} e^{-i} \frac{\partial}{\partial x_i} \tag{2.7}$$

satisfying $\overset{(l)}{\partial}\, \overset{(k)}{z} = \delta^{kl}$. It was shown (Fleury *e al.*, (1995) that most of the theorems of complex analysis can be extended to the $M\mathbb{C}_n$, $n > 2$, spaces. A main property useful for the sequel is that a mapping F: $M\mathbb{C}_n \to M\mathbb{C}_n$ is derivable at z iff

$$\overset{(l)}{\partial}\ \overset{(k)}{F} = 0, \qquad \forall l \neq k \tag{2.8}$$

A function satisfying (2.8) will be called holomorphic.

The lth conjugate $\overset{(l)}{z}$ of z can be simply understood as a tool for calculating $\|z\|$ as in Fleury *et al.* (1995). However, for our convenience, any element z of $M\mathbb{C}_n$ will be seen as parametrized by the n multicomplex number $\overset{(l)}{z}$. This is equivalent to saying, that (2.4) can be inverted. As an easy example, for $n = 2$, a complex number $z = x_0 + x_1 \cdot i$ parametrized by x_0 and x_1 can also be parametrized by the two conjugates $\overset{(0)}{z} = z$ and $\overset{(1)}{z} = \bar{z}$.

This procedure will be useful for developing superspace structure and plausible superconformal fields theories on $M\mathbb{C}_n$. Thus, we give a brief digression on superconformal structure on $\mathbb{C} = M\mathbb{C}_2$. A superconformal structure is the supergeneralization of the conformal one which is equivalent to the usual complex structure of the one-complex dimensional manifold (which is a Riemann surface). More precisely, a superconformal manifold $\hat{S}^{1/N}$ of complex dimension $1/N$ is a real $\mathbf{Z}_2$-graded manifold such that the transition functions on the overlapping $U_\alpha \cap U_\beta$ are the following:

(a) Superanalytic, i.e.,

$$z_\beta = z_\beta\,(z_\alpha, \theta^i_\alpha), \qquad \theta^j_\beta = \theta^j_\beta\,(z_\alpha, \theta^i_\alpha) \tag{2.9}$$

where z_γ, θ^j_γ, $\gamma \in I$, $j = 1, 2, \ldots, N$, and their complex conjugates are the local coordinates of the open set of U_γ.

(b) The usual covariant spinor superderivatives $D_{i\alpha}$, $i = 1.2, \ldots, N$, transform homogeneously

$$D_{i\alpha} = (D_{i\alpha} \cdot \theta^j_\beta)\ D_{j\beta} \tag{2.10}$$

In the $N = 2$ superconformal case, the spinor derivatives D^+ and D^- are the usual supersymmetric operators satisfying the $N = 2$ superalgebra:

$$\{D^+, D^-\} = 2\,\frac{\partial}{\partial z}$$
$$D^{+2} = D^{-2} = 0 \tag{2.11}$$

These equations can be realized locally on the following heterotic superspace:

$$Z^M = \left(z,\ \theta^{\pm} = \frac{1}{\sqrt{2}}\,(\theta^1 \pm i\theta^2)\right)$$
$$D^{\pm} = \frac{\partial}{\partial\theta^{\mp}} + \theta^{\pm}\,\frac{\partial}{\partial z} \tag{2.12}$$

We note here that most of the theorems of the usual complex analysis can be extended to the superspace $SM\mathbb{C}_2^N$, $\sim \hat{S}^{1/N}$ (e.g., Giddingsand and Nelson, 1988; Saidi and Zakkari, 1991, 1992). In the next section, we generalize many of the results on the usual superspace $SM\mathbb{C}_2^N$ to the case of $SM\mathbb{C}_{n>2}^{\hat{N}}$. This latter will be seen to be a set of points parametrized by n real numbers x_i, $i = 0, 1, 2, \ldots, n - 1$, and a set of Grassmann variables to be fixed later on; we recall that n is a power of two.

3. ANALYSIS ON THE SUPERMULTICOMPLEX SPACE $SM\mathbb{C}_{\mathbf{n}}^{\hat{\mathbf{N}}}$

By analogy with the building of the usual superspace $S\mathbb{C}$, we define the superspace $SM\mathbb{C}_n^{\hat{N}}$ as a set of points of $M\mathbb{C}_n$ together with a set of Grassmann variables θ_i. More precisely, a point z in $M\mathbb{C}_n$ is completely determined by its n lth conjugates $\overset{(l)}{z}$ as seen before; thus $SM\mathbb{C}_n^{\hat{N}}$ is the set of points parametrized by the multicomplex numbers $\overset{(l)}{z}$ and the Grassmann coordinates $\overset{(l)}{\theta}_i$, $i = 1, 2, \ldots, N_l$, $l = 0, 1, 2, \ldots, n-1$. Then $\hat{N}$ refers to the set $(N_0, N_1, N_2, \ldots, N_{n-1})$. We note

$$SM\mathbb{C}_n^{\hat{N}} = \{(\overset{(l)}{z}, \overset{(l)}{\theta}_i), \qquad l = 0, 1, 2, \ldots, n-1, i = 1, 2, \ldots, N_l\} \tag{3.1}$$

The motivation of this construction will be examined later. A first remark is that the number of supersymmetries depends on the direction $\overset{(l)}{z}$, in contrast to the usual complex 2D superspace. Indeed, in that case the total superspace splits into two heterotic pieces due to analyticity. In the case of $M\mathbb{C}_n$, this property is given by (2.8) and is manifestly more rich for $n > 2$.)

Recall that the conformal transformations on $M\mathbb{C}_n$: $\overset{(l)}{Z} \to F_l(\overset{(l)}{Z})$ are analytic transformations under which the tangent space elements $e_l \equiv \partial/\partial \overset{(l)}{Z}$ transform as follows:

$$e_l = \sum_{k=0}^{n-1} \frac{\partial F_k}{\partial \overset{(l)}{Z}} \cdot e'_k \tag{3.2}$$

To introduce the notion of superconformal transformations on $SM\mathbb{C}_n^{\hat{N}}$, we first define as in the usual case the superanalyic ones. To this end, we need the covariant spinor derivatives, which read

$$\overset{(l)}{D}_i = \frac{\partial}{\partial \overset{(l)}{\theta}_i} + \overset{(l)}{\theta}_i \frac{\partial}{\partial \overset{(l)}{Z}}, \qquad l = 0, 1, 2, \ldots, n-1, \quad i = 1, 2, \ldots, N_l \tag{3.3}$$

They obviously satisfy

$$\left\{ \overset{(l)}{D}_i, \overset{(k)}{D}_j \right\} = 2\delta_{ij}\,\delta^{kl}\,\partial \overset{(l)}{z} \tag{3.4}$$

A transformation $(\overset{(l)}{z}, \overset{(l)}{\theta}_i) \mapsto f(\overset{(l)}{z}, \overset{(l)}{\theta}_i)$ on $SM\mathbb{C}_n^N$ is said to be superanalytic iff

$$\overset{(k)}{D}_i f(\overset{(l)}{z}, \overset{(l)}{\theta}_i) = 0 \qquad \forall k \neq l \tag{3.5}$$

i.e.,

$$f(\overset{(l)}{z}, \overset{(l)}{\theta}_i) = \left(\overset{(\tilde{l})}{z}(\overset{(l)}{z}, \overset{(l)}{\theta}_i), \overset{(\tilde{l})}{\theta}(\overset{(l)}{z}, \overset{(l)}{\theta}_i) \right), \qquad i = 1, 2, \ldots, N_l \tag{3.6}$$

It is not hard then to check that under a superanalytic transformation the covariant derivatives transform as follows:

$$\overset{(l)}{D}_j = \sum_{i=1}^{N_l} (\overset{(l)}{D}_j \overset{(\tilde{l})}{\theta}_i \overset{(\tilde{l})}{D}_i + (\overset{(l)}{D}_j \overset{(\tilde{l})}{z} - \overset{(\tilde{l})}{\theta}_i \overset{(l)}{D}_j \overset{(\tilde{l})}{\theta}_i\, \partial_{\overset{(\tilde{l})}{z}} \tag{3.7}$$

Superanalytic transformations under which the derivatives $\overset{(l)}{D}_i$ transform homogeneously, i.e.,

$$\overset{(l)}{D}_j = \sum_{i=1}^{N_l} \left(\overset{(l)}{D}_j \overset{(\tilde{l})}{\theta}_i \right) \overset{(\tilde{l})}{D}_i \tag{3.8}$$

are called, as in the usual case, superconformal transformations. So, a superconformal transformation is an analytic one satisfying the superconformal condition on $SM\mathbb{C}_n^{\hat{N}}$:

$$\overset{(l)}{D}_j \overset{(\tilde{l})}{z} = \sum_{i=1}^{N_l} \overset{(\tilde{l})}{\theta}_i \overset{(l)}{D}_j \overset{(\tilde{l})}{\theta}_i \tag{3.9}$$

At this step many results on superconformal manifolds can be generalized to the case of $SM\mathbb{C}_n^{\hat{N}}$; as an example, the measure of Bruce *et al.*, (see Cohm, 1987) reads

$$\overset{(l)}{dZ} = \overset{(l)}{dz} - \overset{(l)}{\theta}_i \, d\overset{(l)}{\theta}_i \tag{3.10}$$

An interesting result consists of the residue theorem on $SM\mathbb{C}_n^N$, which is given by

$$\frac{1}{m!} \underset{z_2}{\overset{m}{\partial}}(k) F(\overset{(k)}{z}_2, \overset{(k)2}{\theta_i}) = \frac{1}{(2\pi e^{n/2})^{n/2}} \int d\overset{(k)1}{z_0} \wedge \cdots \wedge d\overset{(k)1}{z}_{n/2-1} \wedge d\overset{(k)1}{\theta}_1 \wedge \cdots \wedge d\overset{(k)1}{\theta}_{N_k} F(\overset{(k)}{z}_1, \overset{(k)1}{\theta_i}) \frac{\overset{(k)}{\theta}_{12}}{\overset{(k)}{Z}{}_{12}^{m+1}} \tag{3.11}$$

$$\frac{1}{m!} \underset{z_2}{\overset{m}{(k)}} \overset{(k)2}{D_1} F(\overset{(k)}{z}_2, \overset{(k)2}{\theta_i}) = \frac{1}{(2\pi e^{n/2})^{n/2}} \int d\overset{(k)1}{z_0} \wedge \cdots \wedge d\overset{(k)1}{z}_{n/2-1} \wedge d\overset{(k)1}{\theta}_1 \wedge \cdots \wedge d\overset{(k)1}{\theta}_{N_k} F(\overset{(k)}{z}_1, \overset{(k)1}{\theta_1}) \frac{1}{\overset{(k)}{Z}{}_{12}^{m+1}} \tag{3.12}$$

where

$$\overset{(k)2}{D_i} = \partial_{\overset{(k)2}{\theta_i}} + \overset{(k)2}{\theta_i}\, \partial_{\overset{(k)}{z_2}}, \qquad i = 1, 2, \ldots, N_k, \quad k = 0, 1, 2, \ldots, n-1$$

$$\overset{(k)}{\theta}_{12} = \sum_{i=1}^{N_k} \overset{(k)1}{\theta_i} - \overset{(k)2}{\theta_i}$$

$$(\overset{(k)}{z}, \overset{(k)}{\theta_i}) \equiv \left(\overset{(k)}{z}, \overset{(k)}{\theta}_1, \overset{(k)}{\theta}_2, \ldots, \overset{(k)}{\theta}_{N_k} \right)$$

$$\overset{(k)}{z} = \overset{(k)}{z_0} + \overset{(k)}{z_1} + \cdots + \overset{(k)}{z}_{n/2-1}, \qquad z_i = x_i + x_{i+n/2} \cdot e^{n/2}$$

and

$$\overset{(k)}{Z}_{12} = \overset{(k)}{z}_1 - \overset{(k)}{z}_z - \sum_{i=1}^{N_k} \overset{(k)1}{\theta_i}\, \overset{(k)2}{\theta_i}$$

We will not give here a complete proof of (3.11) and (3.12); however, one can verify that they reduce to the right expressions for $n = 2$ and, for example, for $N_0 = N_1 = N = 1$. Indeed, if $n = 2$, then $k = 0$ or 1 and $N = 1$ leads to $i = 1$, so we have

$$\overset{(0)}{D_1} \equiv D \qquad \text{and} \qquad \overset{(1)}{D_1} \equiv \overline{D}$$

We have also

$$\overset{(0)}{\theta}_{12} = \overset{(0)}{\theta}_1 - \overset{(0)}{\theta}_2 \equiv \theta_1 - \theta_2$$

$$\overset{(1)}{\theta}_{12} = \overset{(1)}{\theta}_1 - \overset{(1)}{\theta}_2 \equiv \overline{\theta}_1 - \overline{\theta}_2$$

$$\overset{(0)}{Z}_{12} = \overset{(0)}{z}_1 - \overset{(0)}{z}_2 - \overset{(0)1}{\theta_1}\ \overset{(0)2}{\theta_{1\,1}} \equiv z \equiv z_1 - z_2 - \theta_1\theta_2$$

$$\overset{(1)}{Z}_{12} = \overset{(1)}{z}_1 - \overset{(1)}{z}_2 - \overset{(1)1}{\theta_1}\ \overset{(1)2}{\theta_1} \equiv \overline{z}_1 - \overline{z_2} - \overline{\theta_2}_1\overline{\theta}$$

so that and for $k = 0$ we obtain

$$\frac{1}{m!}\,\partial_{z_2}^m\, F(z_2, \theta_2) = \frac{1}{2\pi i}\int dz_1\, d\theta_1\, F(z_1, \theta_1)\,\frac{\theta_{12}}{Z_{12}^{m+1}}$$

and

$$\frac{1}{m!}\,\partial_{z_2}^m\, DF(z_2, \theta_2) = \frac{1}{2\pi i}\int dz_1 d\theta_1\, F(z_1, \theta_1)\,\frac{1}{Z_{12}^{m+1}} \tag{3.13}$$

These are the known residue formulas for the superspace $S\mathbb{C}^1$; the value $k = 1$ leads to the same expressions with the variables $\overline{z}$ and $\overline{\theta}$.

The expressions (3.11) and (3.12) will be useful in the determination of the operator product expansion (OPE) $J(\overset{(l)}{z_1}, \overset{(l)1}{\theta}{}_i)\ \phi(\overset{(l)}{z_2}, \overset{(l)2}{\theta}{}_i)$ of some supercurrent J and a tensor ϕ $SM\mathbb{C}_n^{\hat{N}}$. We are led then to develop the tensor analysis on a supermulticomplex space. We will introduce superconformal tensors on $SM\mathbb{C}_n^{\hat{N}}$ and discuss a plausible superfield theory on the $SM\mathbb{C}_n^N$ space. For simplicity, we consider the case of $SM\mathbb{C}_n^1$.

Recall first that the primary field $\phi(z)$ on $\mathbb{C}$ is a field which transforms as follows under a conformal transformation $z \to \underset{\sim}{z}(z)$

$$\phi(z) = \left(\frac{d\underset{\sim}{z}}{dz}\right)^h \cdot \tilde{\phi}(\underset{\sim}{z}) \tag{3.14}$$

h is said to be the conformal weight of ϕ.

Equation (3.14) is equivalent to saying that $\phi(z)\ dz^h$ is invariant under any conformal transformation. Following this strategy, the superanalyticity property (3.6) leads us to consider the $SM\mathbb{C}_n^N$ space as a gathering of n "chiral" superspaces $\overset{(l)}{\mathbf{Z}} \equiv (\overset{(l)}{z}, \overset{(l}{\theta}_1, \ldots, \overset{(l)}{\theta}_{N_l})$, $l = 0, 1, 2, \ldots, n - 1$.

A superfield ϕ on $SM\mathbb{C}_n^N$ depends then on the variable $\mathbf{Z}$ for a given l:

$$\phi(\overset{(l)}{z}, \overset{(l)}{\theta}_1, \ldots, \overset{(l)}{\theta}_{N_l}) = \phi(\overset{(l)}{Z})$$

The generalization of (3.14) to the case of the supertensor $\phi_{i_1 \ldots i_N}(\overset{(l)}{\mathbf{Z}})$, $N \geq 0$, leads to

$$\tilde{\phi}_{i_1\ldots i_N}(\overset{(\tilde{l})}{\mathbf{Z}})\; d\overset{(\tilde{l})}{\xi}_{i}{}^{1} \cdots d\overset{(\tilde{l})}{\xi}_{i_N} = \phi_{i_1\ldots i_N}(\overset{(\tilde{l})}{\mathbf{Z}})\, d\overset{(l)}{\xi}_{i_1} \cdots d\overset{(l)}{\xi}_{i_N} \tag{3.15}$$

where the differentials $d\overset{(l)}{\xi}_1$, are generalizations of $d\overset{(l)}{Z}$. They are characterized by the transformation law

$$d\overset{(\tilde{l})}{\xi}_i = \sum_{j=1}^{N^l} (\overset{(l)}{D_j}\, \overset{(\tilde{l})}{\theta}_i)\, d\overset{(l)}{\xi}_j \tag{3.16}$$

under a superconformal transformation $\overset{(l)}{\mathbf{Z}} \to \overset{(\tilde{l})}{\mathbf{Z}}$.

In the next section we develop the ingredients of a supermulticonformal theory on $SM\hat{\mathbb{C}}_n^N$; we will discuss the case $N = 1$ for simplicity.

4. N = 1 SUPERMULTICONFORMAL FIELD THEORY ON $SM\mathbb{C}_n$

As a consequence of the previous discussion, a primary superfield $\phi(\overset{(l)}{\mathbf{Z}})$ on $SM\mathbb{C}_n$ of weight $\overset{(l)}{d}$ is a tensor which transforms as follows under a superconformal transformation $\mathbf{Z} \to \mathbf{Z}$,

$$\tilde{\phi}(\overset{(\tilde{l})}{z}, \overset{(\tilde{l})}{\theta}) = (\overset{(l)}{D}\overset{(l)}{\theta})^{-d}\phi(\overset{(l)}{z}, \overset{(l)}{\theta}) \tag{4.1}$$

Under an infinitesimal superconformal transformation

$$\overset{(l)}{\theta} \mapsto \overset{(\tilde{l})}{\theta} = \delta\overset{(l)}{\theta} \tag{4.2}$$

$$\overset{(l)}{Z} \mapsto \overset{(\tilde{l})}{z} = \overset{(l)}{z} + v\,(\overset{(l)}{z}), \qquad v(\overset{(l)}{Z}, \overset{(l)}{\theta}) = \delta\overset{(l)}{z} + \overset{(l)}{\theta}\,\delta\overset{(l)}{\theta}$$

Equations (4.1) reduces to

$$\delta_v\phi(\overset{(l)}{z}, \overset{(l)}{\theta}) = \left(\frac{1}{2}(\overset{(l)}{D}V\cdot\overset{(l)}{D} + v\partial_{\underset{z}{(l)}} + \overset{(l)}{h}\partial_{\underset{z}{(l)}}v\right)\phi(\overset{(l)}{z}, \overset{(l)}{\theta}) \tag{4.3}$$

where $\overset{(l)}{d} = 2\overset{(l)}{h}$. For the superconformal transformations (4.2), the following relation is valid:

$$\delta_v\phi(\overset{(l)}{z_2}, \overset{(l)2}{\theta}) = \frac{1}{(2\pi e^{n/2})1^{n/2}}\int d\overset{(l)1}{z_0} \wedge \cdots \wedge$$

$$d\overset{(l)1}{z}_{n/2-1}\, d\overset{(l)1}{\theta}\; v(\overset{(l)1}{z}, \overset{(l)1}{\theta})\, T(\overset{(l)}{z}_1, \overset{(l1)}{\theta})\, \phi(\overset{(l)}{z_2}, \overset{(l)2}{\theta}) \tag{4.4}$$

where T is the stress-energy supertensor of the theory. It can be written as

$$T(\overset{(l)}{z}, \overset{(l)}{\theta}) = T_F(\overset{(l)}{Z}) + \overset{(l)}{\theta}\; T_B(\overset{(l)}{z}) \tag{4.5}$$

whereT_F and T_B (respectively the fermionic and bosonic parts of T.

Now by using (3.11) and (3.12), a comparaison of (4.3) and (4.4) leads to the following operator product expansion:

$$T(\overset{(l)}{z}_1, \overset{(l)1}{\theta})\,\Phi(\overset{(l)}{z}_2, \overset{(l)2}{\theta}) = \left[\overset{(l)}{h}\,\frac{\overset{(l)}{\theta}_{12}}{\overset{(l)2}{z_{12}}} + \frac{1/2}{\overset{(l)}{z_{12}}}\overset{(l)}{D}_2 + \frac{\overset{(l)}{\theta}}{\overset{(l)}{z_{12}}}\,\partial^{2(l)}_{z_2}\right]\phi(\overset{(l)}{z}_2, \overset{(l)2}{\theta}) \tag{4.6}$$

Notice that the above equation means that the energy tensor is of weight 3/2, as for the usual $N = 1$, $n = 2$ theory. Moreover, (4.6) can be taken as the defining property of primary superfields on $SM\mathbb{C}^1_n$. We are now interested in the superconformal properties of the superfield T, which is in turn a quasi-primary superfield. We have the following OPE:

$$T(\overset{(l)}{z}_1, \overset{(l)1}{\theta})\,T(\overset{(l)}{z}_2, \overset{(l)2}{\theta}) = \frac{\overset{(l)}{c}/4}{\overset{(l)3}{z_{12}}} + \left[\frac{3}{2}\,\frac{\overset{(l)}{\theta}_{12}}{\overset{(l)2}{z_{12}}} + \frac{1/2}{\overset{(l)}{z_{12}}}\overset{(l)}{D}_2 + \frac{\overset{(l)}{\theta}_{12}}{\overset{(l)}{z_{12}}}\,\partial^{(l)}_{z_2}\right]T(\overset{(l)}{z}_2, \overset{(l)2}{\theta}) \tag{4.7}$$

where the multi-complex number $\overset{(l)}{c}$ stands for a parameter of the theory. Using then the decomposition (4.5) of T into its fermionic and bosonic parts T_F and T_B, the OPE (4.7) leads to the following relations:

$$\begin{aligned}
T_F(\overset{(l)}{z_1})\; T_F(\overset{(l)}{z_2}) &= \frac{\overset{(l)}{c}/4}{\overset{(l)3}{z_{12}}} + \frac{1/2}{\overset{(l)}{z}_{12}}\; T_B(\overset{(l)}{z_2}) \\
T_B(\overset{(l)}{z_1})\; T_F(\overset{(l)}{z_2}) &= \frac{3/2}{\overset{(l)2}{z_{12}}}\; T_F(\overset{(l)}{z}) + \frac{1}{\overset{(l)}{z_{12}}}\,\partial^{(l)}_{z2}\; T_F(\overset{(l)}{z_2})
\end{aligned} \tag{4.8}$$

$$T_B(\overset{(l)}{z}_1)\; T_B(\overset{(l)}{z_2}) = \frac{2}{\overset{(l)2}{z_{12}}}\; T_B(\overset{(l)}{z_2}) + \frac{1}{\overset{(l)}{z_{12}}}\,\partial_{\overset{(l)}{z_2}}\; T_B(\overset{(l)}{z_2}) + \frac{3/4\,\overset{(l)}{c}}{\overset{(l)4}{z_{12}}}\quad \overset{(l)}{z_{12}} = \overset{(l)}{z}_1 - \overset{(l)}{z_2}$$

For invertible z, the Laurent modes of T_F and T_B, say $\overset{(l)}{G}_m$ and $\overset{(l)}{L}_m$, $m \in \mathbf{Z}$, satisfy the following commutation relations:

$$[\overset{(L)}{L}_m, \overset{(l)}{l}_n] = (m-n)\overset{(l)}{L}_{m+n} + \frac{\overset{(l)}{c}}{8}(m^3-m)\delta_{m+n}$$

$$[\overset{(l)}{L}_m, \overset{(l)}{G}_n] = \left(\frac{m}{2}-n\right)\overset{(l)}{G}_{m+n} \tag{4.9}$$

$$\{\overset{(l)}{G}_m, \overset{(l)}{G}_n\} = \overset{(l)}{c}\left(m^2-\frac{1}{4}\right)\delta_{r+n} + 2\overset{(l)}{L}_{m+r}$$

Thus, we have n commuting copies of the $N = 1$ super-Virasoro algebra. The reason why the symmetry algebra becomes infinite dimensional is due to the fact that we are in a critical-dimensional space. In the next section, we give a model of a bosonic theory on $M\mathbb{C}_n$; a fermionic model (and thus a supersymmetric model) needs further developement.

5. BOSONIC THEORY ON $M\mathbb{C}_n$

Recall that a two-dimensional free bosonic field theory is described by the following action:

$$S\mathbb{C} \sim \int dz\, d\bar{z}\, h^{\alpha\beta}\, \partial_\alpha X\, \partial_\beta X, \qquad \alpha, \beta = z, \bar{z} \tag{5.1}$$

with $h^{z\bar{z}} = h^{\bar{z}z} = 1/2$ and $h^{zz} = h^{\bar{z}\bar{z}} = 0$. The equation of motion shows that the field $X(z, \bar{z})$ splits into two parts

$$X(z, \bar{z}) = x(z) + \bar{x}(\bar{z}) \tag{5.2}$$

such that

$$\partial_{\bar{z}} x(z) = \partial_z \bar{x}(\bar{z}) = 0 \tag{5.3}$$

The generalization of (5.1) to the $M\mathbb{C}_n$ spares reads as follows:

$$S_{M\mathbb{C}_n} \sim \int d\overset{(0)}{z} \wedge d\overset{(1)}{z} \wedge \ldots \wedge d\overset{(n-1)}{z}\; J^{kl}\, \partial_k X\, \partial_l X \tag{5.4}$$

where

$$(J^{kl}) = \begin{pmatrix} 0 & & 1/n \\ & \cdot^{\cdot^{\cdot}} & \\ 1/n & & 0 \end{pmatrix} \tag{5.5}$$

The equation of motion leads to the relation

$$J^{kl}\partial_k\partial_l X = 0 \tag{5.6}$$

which obviousely reduces to the known one for $n = 2$ as j^{kl} goes to $\left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$. The solution of (5.6) is given by

$$X(\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}) = \overset{(0)}{x}(\overset{(0)}{z}) + \overset{(1)}{x}(\overset{(1)}{z}) + \ldots + \overset{(n-1)}{x}(\overset{(n-1)}{z}) \tag{5.7}$$

generalizing then (5.2) on $M\mathbb{C}_2$. One can then develop all the machinery of conformal field theory in each direction $\overset{(l)}{z}$, $l = 0, 1, 2, \ldots, n-1$. Finally, we note that (5.4) is a simple generalization of (5.1).

Indeed, the general form of the action on $M\mathbb{C}_n$ is

$$S_{M\mathbb{C}_n} \sim \int d^n s \cdot J^{kl} \partial_k X \partial_l X \tag{5.8}$$

where $d^n s$ is some measure on $M\mathbb{C}_n$ given by

$$d^n s = g_{i_0 i_1 \ldots i_{n-1}} \, d\overset{(j_0)}{z} \wedge \cdots \wedge d\overset{(i_{n-1})}{z} \tag{5.9}$$

The tensors $g_{i_0 i_1 \ldots i_{n-1}}$ and J^{kl} are then shown to satisfy a certain number, of which (5.4) is in fact a plausible solution. We will study of action (5.8) on a future occasion.

Finally, note that we have seen that most of the properties on a multicomplex space can be extended to the associated superspace, called the supermulticomplex space. For this we have adopted a geometric method which is different from the algebraic one using generalized Clifford algebra (Fleury *et al.*, 1995).

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